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An Elementary $U(2)$ Theory of Radiation Fields with Spin J [1]

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Abstract

An elementary theory for a radiation field with any spin J is presented. This is a natural extension of Maxwell's equations for the electromagnetic field. The idea is to use the generators for the $U(2)$ group in a multidimensional representation. These generators are a linear combination of the ones for infinitesimal Lorentz transformations. The constants of the motion in this formalism are discussed. As an example, angular distributions of the Poynting vector are given.

1. *Introduction*

Since the late 1950's R. H. Good, Jr. and his collaborators (Good, 1957; Hammer and Good, 1957, 1958) have developed a field theory for massless particles with any spin J, and later they applied the same idea for massive particles (Weaver *et al.*, 1964; Shay *et al.*, 1965; Nelson and Good, 1968). Their basic idea is to reformulate Maxwell's equations using a spin operator. We will use the same idea and reformulate Maxwell's equations using a genaralized Pauli spin operator instead of the usual spin operator. This idea was originally presented by O. Laporte and G. E. Uhlenbeck (1931). In this paper we wish to present an elementary theory for a radiation field with any spin J. This theory is a natural extension of Maxwell's equations for the electromagnetic field. In this formulation matrix multiplication is the only mathematical technique needed.

The essential idea is to use the generators for the $U(2)$ group in a multidimensional representation whose dimension is determined by the spin of

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the field in question. Since the number of generators needed for the $U(2)$ group is 4, they can correspond to the (x, y, z, ict) coordinates. We will show that these generators are a linear combination of the generators of infinitesimal Lorentz transformations (Naimark, 1964).[3] The simplest nontrivial representation of the $U(2)$ group is of course two dimensional, which could be the unit matrix and Pauli spin matrix. In this case the equation obtained represents a spinor field. The next smallest dimension is 4 and the equation represents a vector field. Since vector fields have only three independent components, the fourth equation gives the divergence condition on the field. The fact that the dimension of the generators of the $U(2)$ group is 4 in the case of a vector field is consistent with the concept of composite fields. Namely, the vector fields are constructed from two spinor fields.

The same idea can be extended to fields with any spin J that are constructed from a spinor and a $(J - \frac{1}{2})$ field. The total dimension of the $U(2)$ group is $4J$; the real fields span $(2J + 1)$ dimensions and the remaining $(2J - 1)$ dimensions describe the divergence conditions.

We will start with the classical Maxwell's equations and reformulate them in terms of 4×4 matrices, $\bar{\Sigma}$, which with the unit matrix, constitute the generators of the $U(2)$ group in four-dimensional representation. After examining this formalism, we will be able to find a generalized Pauli spin matrix that is directly related to the generators of infinitesimal Lorentz transformations. Also we will show that in this representation any set of equations representing a field with spin $J \geqslant 1$ can be reduced to exactly the same form as Maxwell's equations. In this case we have to interpret all physical quantities, for example, $E_x, B_x, A_x, \phi, J_x, \rho$ as each having $(2J - 1)$ components. This is possible for both integer and half-odd integer spin fields. Next, we will discuss constants of the motion in the Hamiltonian formalism. In this latter section we will present a fundamental constant of the motion and additional $3 \times 4J^2$ independent constants of the motion. Finally, as an application, we will show the radiation patterns of fields with spin J and the total angular momentum I.

2. *Maxwell's Equations*

Maxwell's equations are given by

$$\begin{aligned} \bar{\nabla} \cdot \bar{E} &= 4\pi\rho \\ \bar{\nabla} \cdot \bar{B} &= 0 \\ \frac{1}{c}\frac{\partial}{\partial t}\bar{E} - \bar{\nabla} \times \bar{B} &= -\frac{4\pi}{c}\bar{j} \\ \frac{1}{c}\frac{\partial}{\partial t}\bar{B} + \bar{\nabla} \times \bar{E} &= 0 \end{aligned} \tag{2.1}$$

[3] The only difference between his definition and ours is the X_4 coordinate. He chooses $X_4 = ct$ while we choose $X_4 = ict$. Therefore b_1, b_2, and b_3 have a different expression. Since we use ia_k and ib_k as A_k and B_k, we obtained the commutation relation (3.2).

where ρ and $\bar{j}$ are charge and current density, respectively, and satisfy the continuity condition

$$\frac{\partial}{\partial t}\rho + \bar{\nabla}\cdot\bar{j} = 0 \tag{2.2}$$

If the quantum-mechanical equivalencies between energy and the time derivative and between momentum and the space derivative are introduced, and the two linear combinations

$$\begin{aligned}\bar{F}^{+} &= \bar{B} + i\bar{E}\\ \bar{F}^{-} &= \bar{B} - i\bar{E}\end{aligned} \tag{2.3}$$

are also introduced, the four equations can be expressed in two separate groups:

$$\bar{p}\cdot\bar{F}^{+} = \frac{4\pi\hbar}{ic}\,ic\rho \qquad -i\frac{\epsilon}{c}\bar{F}^{+} + \bar{p}\times\bar{F}^{+} = \frac{4\pi\hbar}{ic}\bar{j} \tag{2.4}$$

and

$$\bar{p}\cdot\bar{F}^{-} = -\frac{4\pi\hbar}{ic}\,ic\rho \qquad i\frac{\epsilon}{c}\bar{F}^{-} + \bar{p}\times\bar{F}^{-} = \frac{4\pi\hbar}{ic}\bar{j} \tag{2.5}$$

with the continuity condition

$$i\frac{\epsilon}{c}\,ic\rho + \bar{p}\cdot\bar{j} = 0 \tag{2.6}$$

Equations (2.4) and (2.5) can be expressed in the matrix forms

$$\begin{pmatrix} -i\frac{\epsilon}{c} & -pz & py & -px \\ pz & -i\frac{\epsilon}{c} & -px & -py \\ -py & px & -i\frac{\epsilon}{c} & -pz \\ px & py & pz & -i\frac{\epsilon}{c} \end{pmatrix} \begin{pmatrix} Fx^{+} \\ Fy^{+} \\ Fz^{+} \\ 0 \end{pmatrix} = \frac{4\pi\hbar}{ic} \begin{pmatrix} jx \\ jy \\ jz \\ ic\rho \end{pmatrix} \tag{2.7}$$

and

$$\begin{pmatrix} i\frac{\epsilon}{c} & -pz & py & -px \\ pz & i\frac{\epsilon}{c} & -px & -py \\ -py & px & i\frac{\epsilon}{c} & -pz \\ px & py & pz & i\frac{\epsilon}{c} \end{pmatrix} \begin{pmatrix} Fx^{-} \\ Fy^{-} \\ Fz^{-} \\ 0 \end{pmatrix} = \frac{4\pi\hbar}{ic} \begin{pmatrix} jx \\ jy \\ jz \\ -ic\rho \end{pmatrix} \tag{2.8}$$

where these two matrices have determinant $[(\epsilon/c)^2 - p^2]^2$.

From these two equations two wave equations can be obtained:

$$\left[\left(\frac{\epsilon}{c}\right)^2 - p^2\right]\begin{pmatrix} 1\!\!1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} \bar{F}^+ \\ 0 \end{pmatrix} = \frac{4\pi\hbar}{ic}\begin{pmatrix} \bar{p} \times \bar{j} + i\dfrac{\epsilon}{c}\bar{j} - \bar{p}ic\rho \\ \bar{p}\cdot\bar{j} + i\dfrac{\epsilon}{c}ic\rho \end{pmatrix} \tag{2.9}$$

and

$$\left[\left(\frac{\epsilon}{c}\right)^2 - p^2\right]\begin{pmatrix} 1\!\!1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} \bar{F}^- \\ 0 \end{pmatrix} = \frac{4\pi\hbar}{ic}\begin{pmatrix} \bar{p} \times \bar{j} - i\dfrac{\epsilon}{c}\bar{j} + \bar{p}ic\rho \\ \bar{p}\cdot\bar{j} + i\dfrac{\epsilon}{c}ic\rho \end{pmatrix} \tag{2.10}$$

where the fourth components of equations (2.9) and (2.10) represent the continuity condition. Therefore one can consider equations (2.7) and (2.8) as basic equations for the electromagnetic field.

Now three matrices are defined:

$$\Sigma_x \equiv \begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & -i & 0 \\ 0 & i & 0 & 0 \\ i & 0 & 0 & 0 \end{pmatrix}, \Sigma_y \equiv \begin{pmatrix} 0 & 0 & i & 0 \\ 0 & 0 & 0 & -i \\ -i & 0 & 0 & 0 \\ 0 & i & 0 & 0 \end{pmatrix}, \Sigma_z \equiv \begin{pmatrix} 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \end{pmatrix} \tag{2.11}$$

The matrix operators in equations (2.7) and (2.8) can then be expressed in the form (Laporte and Uhlenbeck, 1931)

$$\begin{pmatrix} \mp i\dfrac{\epsilon}{c} & -pz & py & -px \\ pz & \mp i\dfrac{\epsilon}{c} & -px & -py \\ -py & px & \mp i\dfrac{\epsilon}{c} & -pz \\ px & py & pz & \mp i\dfrac{\epsilon}{c} \end{pmatrix} = \mp i\frac{\epsilon}{c}\,1\!\!1 - i\bar{p}\cdot\bar{\Sigma} \tag{2.12}$$

where $1\!\!1$ represents a 4 x 4 unit matrix. As is easily seen, the three matrices are Hermitian and satisfy the relationships

$$\Sigma_i^2 = 1\!\!1, \quad \Sigma_i\Sigma_j = -\Sigma_j\Sigma_i = i\Sigma_k \tag{2.13}$$

where (i, j, k) are cyclic permutations of (x, y, z). These relationships are exactly those satisfied by the Pauli spin matrices. Therefore, the $\bar{\Sigma}$ matrices and unit matrix are considered as the four-dimensional representation of the $U(2)$ group.

Under the unitary proper transformation that corresponds to the transformation from cartesian coordinates to spherical tensor coordinates,

$$U = \begin{pmatrix} -\frac{1}{\sqrt{2}} & \frac{i}{\sqrt{2}} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \frac{1}{\sqrt{2}} & \frac{i}{\sqrt{2}} & 0 & 0 \\ 0 & 0 & 0 & -i \end{pmatrix}, \quad \det(U) = 1 \tag{2.14}$$

the Σ matrices are transformed to

$$\Sigma_x = \begin{pmatrix} 0 & \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} & 0 \\ 0 & \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} & 0 \end{pmatrix}, \quad \Sigma_y = \begin{pmatrix} 0 & -\frac{i}{\sqrt{2}} & 0 & \frac{i}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & 0 & -\frac{i}{\sqrt{2}} & 0 \\ 0 & \frac{i}{\sqrt{2}} & 0 & \frac{i}{\sqrt{2}} \\ -\frac{i}{\sqrt{2}} & 0 & -\frac{i}{\sqrt{2}} & 0 \end{pmatrix},$$

$$\Sigma_z = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \tag{2.15}$$

In these expressions the 4 x 4 matrix will be divided into four parts,

$$\bar{\Sigma} = \begin{pmatrix} \bar{S} & \bar{G} \\ \bar{D} & \bar{0} \end{pmatrix} \tag{2.16}$$

where $\bar{S}$ is the 3 x 3 spin matrix with magnitude $J = 1$, $\bar{D}$ is the 1 x 3 divergence matrix, $\bar{G}$ is the 3 x 1 gradient matrix, and adjoint to $\bar{D}$, and $\bar{0}$ is the 1 x 1 zero matrix (see Appendix). Then the basic equations (2.7) and (2.8) can be expressed as

$$\left[\frac{\epsilon}{c}\begin{pmatrix} \mathbb{1} & 0 \\ 0 & 1 \end{pmatrix} + \bar{p} \cdot \begin{pmatrix} \bar{S} & \bar{G} \\ \bar{D} & \bar{0} \end{pmatrix}\right]\begin{pmatrix} \bar{F}'^{+} \\ 0 \end{pmatrix} = \frac{4\pi\hbar}{c}\begin{pmatrix} \bar{j} \\ c\rho \end{pmatrix} \tag{2.17}$$

and

$$\left[-\frac{\epsilon}{c}\begin{pmatrix}\mathbb{1} & 0\\ 0 & 1\end{pmatrix}+\bar{p}\cdot\begin{pmatrix}\bar{S} & \bar{G}\\ \bar{D} & \bar{0}\end{pmatrix}\right]\begin{pmatrix}\bar{F}'^{-}\\ 0\end{pmatrix}=\frac{4\pi\hbar}{c}\begin{pmatrix}\bar{j}\\ -c\rho\end{pmatrix} \qquad (2.18)$$

where $\bar{F}'^{\pm} = U\bar{F}^{\pm}$.

3. *The Generalized $\bar{\Sigma}$ Matrix and the Infinitesimal Lorentz Group*

The previous $\bar{\Sigma}$ matrix will be divided into two operators

$$A_i = \begin{pmatrix}S_i & 0\\ 0 & 0_i\end{pmatrix}, \quad B_i = \begin{pmatrix}0 & G_i\\ D_i & 0\end{pmatrix} \qquad (3.1)$$

These six matrices satisfy the commutation relations

$$\begin{aligned}[][A_i, B_i] &= 0\\ [A_i, A_j] &= [B_i, B_j] = iA_k\\ [A_i, B_j] &= [B_i, A_j] = iB_k\end{aligned} \qquad (3.2)$$

where (i, j, k) are cyclic permutations of (x, y, z). From these relationships one can see that all A_i and B_i are the generators of infinitesimal Lorentz transformations except for a phase factor i (Naimark, 1964). Since $\Sigma_i = A_i + B_i$ one can define another combination $\Sigma_i' = A_i - B_i$ and then one can separate Σ_i and Σ_i' completely, namely,

$$[\Sigma_i, \Sigma'_{i'}] = 0$$

for all i and i', and

$$\begin{aligned}[][\Sigma_i, \Sigma_j] &= 2i\Sigma_k\\ [\Sigma_i', \Sigma_j'] &= 2i\Sigma_k'\end{aligned} \qquad (3.3)$$

for cyclic permutations of (x, y, z).

In this paper we have used Σ_i in equations (2.17) and (2.18), but one can express the same equations in terms of Σ_i' if source $(\bar{j}, c\rho)$ are interchanged. Therefore, selection of Σ_i rather than Σ_i' is ambiguous at the moment, but the reason for the choice will be given in Section 6.

A successful generalized $\bar{\Sigma}(J)$ matrix for a spin-J field, analogous to the definition (2.16) is

$$\bar{\Sigma}(J) = \frac{1}{J}\begin{pmatrix}\bar{S}(J) & \bar{G}(J)\\ \bar{D}(J) & -\bar{S}(J-1)\end{pmatrix} \qquad (3.4)$$

where $\bar{\Sigma}(J)$ is the $4J \times 4J$ matrix and are required to satisfy the relationships

$$\Sigma_i(J)^2 = \mathbb{1}, \qquad \Sigma_i(J)\Sigma_j(J) = -\Sigma_j(J)\Sigma_i(J) = i\Sigma_k(J) \qquad (3.5)$$

Also $\bar{S}(J)$ is the $(2J+1) \times (2J+1)$ spin matrix with magnitude J, $\bar{D}(J)$ is the $(2J-1) \times (2J+1)$ divergence matrix, $\bar{G}(J)$ is the $(2J+1) \times (2J-1)$ gradient matrix, and adjoint to $\bar{D}(J)$ and $\bar{S}(J-1)$ is the $(2J-1) \times (2J-1)$ spin matrix with magnitude $J-1$. From the relationships (3.5) one obtains the following relationships:

$$
\begin{aligned}
S_i(J)S_i(J) + G_i(J)D_i(J) &= J^2 \mathbb{1}(J) \\
S_i(J)G_i(J) - G_i(J)S_i(J-1) &= 0 \\
D_i(J)S_i(J) - S_i(J-1)D_i(J) &= 0 \\
D_i(J)G_i(J) + S_i(J-1)S_i(J-1) &= J^2 \mathbb{1}(J-1) \\
S_i(J)S_j(J) + G_i(J)D_j(J) &= -S_j(J)S_i(J) - G_j(J)D_i(J) \\
&= iJS_k(J) \\
S_i(J)G_j(J) - G_i(J)S_j(J-1) &= -S_j(J)G_i(J) + G_j(J)S_i(J-1) \\
&= iJG_k(J) \\
D_i(J)S_j(J) - S_i(J-1)D_j(J) &= -D_j(J)S_i(J) + S_j(J-1)D_i(J) \\
&= iJD_k(J) \\
D_i(J)G_j(J) + S_i(J-1)S_j(J-1) &= -D_j(J)G_i(J) - S_j(J-1)S_i(J-1) \\
&= -iJS_k(J-1)
\end{aligned}
\tag{3.6}
$$

One can easily show that these are consistent with the relationships

$$
\begin{aligned}
S_i(J)S_j(J) - S_j(J)S_i(J) &= iS_k(J) \\
S_i(J-1)S_j(J-1) - S_j(J-1)S_i(J-1) &= iS_k(J-1) \\
S_i(J)G_j(J) - G_j(J)S_i(J-1) &= -S_j(J)G_i(J) + G_i(J)S_j(J-1) \\
&= iG_k(J) \\
D_i(J)S_j(J) - S_j(J-1)D_i(J) &= -D_j(J)S_i(J) + S_i(J-1)D_j(J) \\
&= iD_k(J)
\end{aligned}
\tag{3.7}
$$

where (i, j, k) are cyclic orders of (x, y, z).

Analogous to the definition (3.1) one can define

$$
A_i(J) = \begin{pmatrix} S_i(J) & 0 \\ 0 & S_i(J-1) \end{pmatrix}, \quad B_i(J) = \begin{pmatrix} \left(\frac{1}{J} - 1\right) S_i(J) & \frac{1}{J} G_i(J) \\ \frac{1}{J} D_i(J) & -\left(\frac{1}{J} + 1\right) S_i(J-1) \end{pmatrix}
\tag{3.8}
$$

These six $A_i(J)$ and $B_i(J)$ satisfy the relationship (3.2). Under a unitary proper transformation, these $A_i(J)$ and $B_i(J)$ are transformed into

$$A_i(J) = \tfrac{1}{2}\Sigma_i^p(J) + \tfrac{1}{2}\Sigma_i'^p(J) \tag{3.9}$$

and

$$B_i(J) = \tfrac{1}{2}\Sigma_i^p(J) - \tfrac{1}{2}\Sigma_i'^p(J)$$

where

$$\Sigma_x^p(J) = \begin{pmatrix} 0 & \mathbb{1}(J-\frac{1}{2}) \\ \mathbb{1}(J-\frac{1}{2}) & 0 \end{pmatrix}, \quad \Sigma_y^p(J) = \begin{pmatrix} 0 & -i\mathbb{1}(J-\frac{1}{2}) \\ i\mathbb{1}(J-\frac{1}{2}) & 0 \end{pmatrix},$$

$$\Sigma_z^p(J) = \begin{pmatrix} \mathbb{1}(J-\frac{1}{2}) & 0 \\ 0 & -\mathbb{1}(J-\frac{1}{2}) \end{pmatrix} \tag{3.10}$$

and

$$\Sigma_i'^p(J) = 2\begin{pmatrix} S_i(J-\frac{1}{2}) & 0 \\ 0 & S_i(J-\frac{1}{2}) \end{pmatrix} \tag{3.11}$$

If $J = \frac{1}{2}, A_i(\frac{1}{2}) = B_i(\frac{1}{2}) = \frac{1}{2}\sigma_i$, hence $\Sigma_i(\frac{1}{2}) = \sigma_i$ and $\Sigma_i'(\frac{1}{2}) = 0$ where σ_i are Pauli spin matrixes. We will discuss the unitary proper transformation in Section 6.

From relationship (3.6) one can derive the generalized vector algebra for a multidimensional representation (see Appendix).

4. *Generalized Maxwell's Equations*

Analogous to the two basic Maxwell's equations (2.17) and (2.18), one can construct two basic field equations for spin-J fields,

$$\left[\frac{\epsilon}{c}\begin{pmatrix} \mathbb{1}(J) & 0 \\ 0 & \mathbb{1}(J-1) \end{pmatrix} + \bar{p}\cdot\frac{1}{J}\begin{pmatrix} \bar{S}(J) & \bar{G}(J) \\ \bar{D}(J) & -\bar{S}(J-\ ) \end{pmatrix}\right]\begin{pmatrix} F^+(J) \\ 0(J-1) \end{pmatrix} = \frac{4\kappa\hbar}{c}\begin{pmatrix} j(J) \\ c\rho(J-1) \end{pmatrix} \tag{4.1}$$

and

$$\left[-\frac{\epsilon}{c}\begin{pmatrix} \mathbb{1}(J) & 0 \\ 0 & \mathbb{1}(J-1) \end{pmatrix} + \bar{p}\cdot\frac{1}{J}\begin{pmatrix} \bar{S}(J) & \bar{G}(J) \\ \bar{D}(J) & -\bar{S}(J-1) \end{pmatrix}\right]\begin{pmatrix} F^-(J) \\ 0(J-1) \end{pmatrix} = \frac{4\pi\hbar}{c}\begin{pmatrix} j(J) \\ -c\rho(J-1) \end{pmatrix} \tag{4.2}$$

where $F^{\pm}(J)$ are $(2J+1)$-component fields and $0(J-1)$ are $(2J-1)$-component null fields. The corresponding wave equations are

$$\left[-\left(\frac{\epsilon}{c}\right)^2+p^2\right]\begin{pmatrix}\mathbb{1}(J) & 0\\ 0 & \mathbb{1}(J-1)\end{pmatrix}\begin{pmatrix}F^{+}(J)\\ 0(J-1)\end{pmatrix}=\frac{4\pi\hbar}{c}$$
$$\times\left[-\frac{\epsilon}{c}\begin{pmatrix}\mathbb{1}(J) & 0\\ 0 & \mathbb{1}(J-1)\end{pmatrix}+\bar{p}\cdot\frac{1}{J}\begin{pmatrix}\bar{S}(J) & \bar{G}(J)\\ \bar{D}(J) & -\bar{S}(J-1)\end{pmatrix}\right]\begin{pmatrix}j(J)\\ c\rho(J-1)\end{pmatrix} \tag{4.3}$$

and

$$\left[-\left(\frac{\epsilon}{c}\right)^2+p^2\right]\begin{pmatrix}\mathbb{1}(J) & 0\\ 0 & \mathbb{1}(J-1)\end{pmatrix}\begin{pmatrix}F^{-}(J)\\ 0(J-1)\end{pmatrix}=\frac{4\pi\hbar}{c}$$
$$\times\left[\frac{\epsilon}{c}\begin{pmatrix}\mathbb{1}(J) & 0\\ 0 & \mathbb{1}(J-1)\end{pmatrix}+\bar{p}\cdot\frac{1}{J}\begin{pmatrix}\bar{S}(J) & \bar{G}(J)\\ \bar{D}(J) & -\bar{S}(J-1)\end{pmatrix}\right]\begin{pmatrix}j(J)\\ -c\rho(J-1)\end{pmatrix} \tag{4.4}$$

Generalized continuity conditions are expressed as

$$\begin{aligned} i\frac{\epsilon}{c}\mathbb{1}(J-1)ic\rho(J-1)+\bar{p}\cdot\frac{1}{J}\bar{D}(J)j(J)&=0(J-1)\\ \bar{p}\cdot\frac{1}{J}\bar{S}(J-1)ic\rho(J-1)&=0(J-1)\end{aligned} \tag{4.5}$$

The first one is an ordinary continuity condition while the second one represents an irrotational property of charge density in a generalized form.

Since $F_x^{\pm}=D_xF^{\pm}$ holds for a vector field, one can define, analogous to it, a multicomponent vector and scalar as

$$\begin{aligned}
\bar{F}^{+}&\equiv\frac{1}{J}\bar{D}(J)F^{+}(J)\\
\bar{F}^{+}&\equiv\frac{1}{J}\bar{D}(J)F^{-}(J)\\
\bar{j}^{+}&\equiv\frac{1}{J}\bar{D}(J)j(J)+i\frac{1}{J}\bar{S}(J-1)ic\rho(J-1)\\
\bar{j}^{-}&\equiv\frac{1}{J}\bar{D}(J)j(J)-i\frac{1}{J}\bar{S}(J-1)ic\rho(J-1)\\
ic\rho&\equiv ic\rho(J-1)
\end{aligned} \tag{4.6}$$

where all quantities have $(2J-1)$ components. Then, with the help of relationships (3.6), one can derive the equations

$$\bar{p}\cdot\bar{F}^{+}=+\frac{4\pi\hbar}{ic}ic\rho \qquad -i\frac{\epsilon}{c}\bar{F}^{+}+\bar{p}\times\bar{F}^{+}=\frac{4\pi\hbar}{ic}\bar{j}^{+} \tag{4.7}$$

and

$$\bar{p}\cdot\bar{F}^- = -\frac{4\pi\hbar}{ic}ic\rho \quad i\frac{\epsilon}{c}\bar{F}^- + \bar{p}\times\bar{F}^- = \frac{4\pi\hbar}{ic}\,\bar{j}^- \tag{4.8}$$

The continuity condition is expressed as

$$i\frac{\epsilon}{c}ic\rho + \bar{p}\cdot\bar{j}^{\pm} = 0 \tag{4.9}$$

These equations are exactly the same as equations (2.4), (2.5), and (2.6) except for current densities $j^{\pm}$.

By exactly the same method, one can define potentials:

$$\begin{pmatrix} F^+(J) \\ 0(J-1) \end{pmatrix} = \frac{1}{\hbar}\left[-\frac{\epsilon}{c}\begin{pmatrix} \mathbb{1}(J) & 0 \\ 0 & \mathbb{1}(J-1) \end{pmatrix} + \bar{p}\cdot\frac{1}{J}\begin{pmatrix} \bar{S}(J) & \bar{G}(J) \\ \bar{D}(J) & -\bar{S}(J-1) \end{pmatrix}\right]\begin{pmatrix} A(J) \\ \phi(J-1) \end{pmatrix} \tag{4.10}$$

and

$$\begin{pmatrix} F^-(J) \\ 0(J-1) \end{pmatrix} = \frac{1}{\hbar}\left[\frac{\epsilon}{c}\begin{pmatrix} \mathbb{1}(J) & 0 \\ 0 & \mathbb{1}(J-1) \end{pmatrix} + \bar{p}\cdot\frac{1}{J}\begin{pmatrix} \bar{S}(J) & \bar{G}(J) \\ \bar{D}(J) & -\bar{S}(J-1) \end{pmatrix}\right]\begin{pmatrix} A(J) \\ -\phi(J-1) \end{pmatrix} \tag{4.11}$$

The generalized Lorentz conditions are then

$$i\frac{\epsilon}{c}\mathbb{1}(J-1)i\phi(J-1) + \bar{p}\cdot\frac{1}{J}\bar{D}(J)A(J) = 0(J-1) \tag{4.12}$$

and

$$\bar{p}\cdot\frac{1}{J}\bar{S}(J-1)i\phi(J-1) = 0(J-1)$$

If we define

$$\begin{aligned} \bar{A}^+ &\equiv \frac{1}{J}D(J)A(J) + i\frac{1}{J}\bar{S}(J-1)i\phi(J-1) \\ \bar{A}^- &\equiv \frac{1}{J}\bar{D}(J)A(J) - i\frac{1}{J}\bar{S}(J-1)i\phi(J-1) \\ i\phi &\equiv i\phi(J-1) \end{aligned} \tag{4.13}$$

one can easily obtain

$$\begin{aligned} \bar{F}^+ &= \frac{i}{\hbar}\left[\bar{p}\times\bar{A}^+ + i\frac{\epsilon}{c}\bar{A}^+ - \bar{p}i\phi\right] \\ \bar{F}^- &= \frac{i}{\hbar}\left[\bar{p}\times\bar{A}^- - i\frac{\epsilon}{c}\bar{A}^- + \bar{p}i\phi\right] \end{aligned} \tag{4.14}$$

and the Lorentz condition

$$i\frac{\epsilon}{c}i\phi + \bar{p}\cdot\bar{A}^{\pm} = 0 \qquad (4.15)$$

Equations (4.14) and (4.15) are exactly the same as those of the electromagnetic fields except for $\bar{A}^{\pm}$.

5. *Hamiltonian and Constants of Motion*

Using equations (4.1) and (4.2) one can write

$$\epsilon\begin{pmatrix}F^{+}(J)\\ 0(J-1)\end{pmatrix} = -c\bar{p}\cdot\bar{\Sigma}(J)\begin{pmatrix}F^{+}(J)\\ 0(J-1)\end{pmatrix} + 4\pi\hbar\begin{pmatrix}j(J)\\ c\rho(J-1)\end{pmatrix} \qquad (5.1)$$

and

$$\epsilon\begin{pmatrix}F^{-}(J)\\ 0(J-1)\end{pmatrix} = c\bar{p}\cdot\bar{\Sigma}(J)\begin{pmatrix}F^{-}(J)\\ 0(J-1)\end{pmatrix} - 4\pi\hbar\begin{pmatrix}j(J)\\ -c\rho(J-1)\end{pmatrix} \qquad (5.2)$$

Defining

$$\psi^{+}(J) \equiv \begin{pmatrix}F^{+}(J)\\ 0(J-1)\end{pmatrix}, \qquad \psi^{-}(J) \equiv \begin{pmatrix}F^{-}(J)\\ 0(J-1)\end{pmatrix} \qquad (5.3)$$

it is found that for a no-source region the above equations become

$$\epsilon\psi^{\pm}(J) = \mp c\bar{p}\cdot\bar{\Sigma}(J)\psi^{\pm}(J) \qquad (5.4)$$

One can interpret $H_0^{\pm} = \mp c\bar{p}\cdot\bar{\Sigma}(J)$ as Hamiltonians for free $\psi^{\pm}(J)$ fields.

Now one can see that the total angular momentum is a good quantum number. The total angular momentum $\bar{I}(J)$ is defined as

$$\bar{I}(J) = \bar{L} + \hbar\begin{pmatrix}\bar{S}(J) & 0\\ 0 & \bar{S}(J-1)\end{pmatrix} \qquad (5.5)$$

where L represents orbital angular momentum. One can easily prove that

$$[\bar{I}(J), H_0^{\pm}(J)] = 0 \qquad (5\ 6)$$

Another quantity which also commutes with the free-field Hamiltonian is

$$\bar{I}^{p}(J) = \bar{L} + \tfrac{1}{2}\hbar\bar{\Sigma}(J) \qquad (5.7)$$

Since the $\bar{\Sigma}$ matrix satisfies the relationship (3.5), it is easily seen that

$$[\bar{I}^{p}(J), H_0^{\pm}(J)] = 0 \qquad (5.8)$$

Since there is no unitary transformation between $\bar{I}(J)$ and $\bar{I}^{p}(J)$ they are two different quantities.

In order to obtain a relation between them, a unitary transformation $U'(J)$ will be defined to satisfy

$$U'(J)\Sigma_i(J)U'(J)^{-1} = \Sigma_i^p(J) \tag{5.9}$$

where the $\Sigma_i^p(J)$ were defined in equation (3.10). If this unitary transformation is applied to the spin matrix in equation (5.5), it is found that

$$U'(J)\begin{pmatrix}\bar{S}(J) & 0 \\ 0 & \bar{S}(J-1)\end{pmatrix} U'(J)^{-1} = \tfrac{1}{2}\bar{\Sigma}^p(J) + \begin{pmatrix}\bar{S}(J-\frac{1}{2}) & 0 \\ 0 & \bar{S}(J-\frac{1}{2})\end{pmatrix} \tag{5.10}$$

This is the same matrix $A_i(J)$ of the Lorentz group defined in equation (3.8). Therefore one can prove the relationship between equations (5.6) and (5.8) as

$$\begin{aligned}\left[\begin{pmatrix}\bar{S}(J) & 0 \\ 0 & \bar{S}(J-1)\end{pmatrix}, \bar{p}\cdot\bar{\Sigma}(J)\right] &= U'(J)^{-1}U'(J)\left[\begin{pmatrix}\bar{S}(J) & 0 \\ 0 & \bar{S}(J-1)\end{pmatrix}, \bar{p}\cdot\bar{\Sigma}(J)\right] \\ &\quad \times U'(J)^{-1}U'(J) = U'(J)^{-1}\left[\tfrac{1}{2}\bar{\Sigma}^p(J), \bar{p}\cdot\bar{\Sigma}^p(J)\right] \\ &\quad \times U'(J) + U'(J)^{-1}\left[\begin{pmatrix}\bar{S}(J-\frac{1}{2}) & 0 \\ 0 & \bar{S}(J-\frac{1}{2})\end{pmatrix}\right. \\ &\quad \left.\times \bar{p}\cdot\bar{\Sigma}^p(J)\right] U'(J)\end{aligned} \tag{5.11}$$

and since $\Sigma_i^p(J)$ has the form shown in equations (3.10), it is obvious that the second term of the equation (5.11) is zero. Therefore

$$\left[\begin{pmatrix}\bar{S}(J) & 0 \\ 0 & \bar{S}(J-1)\end{pmatrix}, \bar{p}\cdot\bar{\Sigma}(J)\right] = \left[\tfrac{1}{2}\bar{\Sigma}(J), \bar{p}\cdot\bar{\Sigma}(J)\right] \tag{5.12}$$

The commutation relation between the spin operator and the free-field Hamiltonian is equivalent to that between the $\frac{1}{2}\Sigma(J)$ operator and the free-field Hamiltonian.

From the above argument one can construct any constant of motion by

$$\bar{K}(J) = U'(J)^{-1}\left[\begin{pmatrix}\bar{X}(J-\frac{1}{2}) & 0 \\ 0 & \bar{X}(J-\frac{1}{2})\end{pmatrix} + \tfrac{1}{2}\bar{\Sigma}^p(J)\right] U'(J) \tag{5.13}$$

where $\bar{L} + \hbar\bar{K}(J)$ is always commutable with $H_0^{\pm}(J)$.

In conclusion $\bar{I}^p(J)$ is a fundamental constant of motion because it corresponds to $\bar{X}(J-\frac{1}{2}) = \bar{0}(J-\frac{1}{2})$ while the total angular momentum is one of induced constant of motion. Except for a multiplying constant, one can easily find $4J^2$ independent $X(J-1/2)$. This result is entirely different from the one obtained in the ordinary spin formalism, where the total angular momentum is the only constant of motion.

6. *Meaning of $U'(J)$ and Composite Fields*

The meaning of the unitary transformation $U'(J)$ will be clear if one considers an explicit example. For a field with spin $J = 1$, the unitary *proper* transformation is

$$U'(L) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \\ 0 & 0 & 1 & 0 \end{pmatrix}, \quad \det(U') = 1 \tag{6.1}$$

and if one operates with $U'(1)^{-1}$ on a composite field consisting of two spinor fields, $U_\alpha U_\beta$, one obtains

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 0 & 1 \\ 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \end{pmatrix} \begin{pmatrix} U_\uparrow(1)\, U_\uparrow(2) \\ U_\uparrow(1)\, U_\downarrow(2) \\ U_\downarrow(1)\, U_\uparrow(2) \\ U_\downarrow(1)\, U_\downarrow(2) \end{pmatrix} = \begin{pmatrix} U_{11}\,(1,2) \\ U_{10}\,(1,2) \\ U_{1\,-1}\,(1,2) \\ U_{00}\,(1,2) \end{pmatrix} \tag{6.2}$$

A set of the first three fields corresponds to a field with spin 1 and the last one corresponds to a field with spin 0.

From the above example one obtains for elements of the $U'(1)^{-1}$ matrix Glebsch-Gorden coefficients, that is,

$$(U^*_{\lambda,\beta}(1,2), U_\alpha(1)\, U_\beta(2)) = (\tfrac{1}{2}, \alpha, \tfrac{1}{2}, \beta \,|\, \lambda, \mu) \tag{6.3}$$

Hence the unitary transformation $U'(J)$ is a proper transformation connecting a composite space of a spinor $J_1 = \frac{1}{2}$ and a field with $J_2 = J - \frac{1}{2}$, with an irreducible spherical tensor space of fields with spin J and $J - 1$. This is the reason why a field with spin J always has associated with it $(J - 1)$ divergence conditions.

The $\bar{\Sigma}$ matrix that was defined in equation (2.11) is obtained with the *proper* transformation

$$\bar{\Sigma} = U'(1)^{-1} \bar{\Sigma}^p(1) U'(1) \tag{6.4}$$

In order to obtain $\bar{\Sigma}'$, one has to use an *improper* transformation. This is the reason why we have used $\bar{\Sigma}$ instease of $\bar{\Sigma}'$ in the fundamental equations (2.17) and (2.18).

Using the above method, one can easily find matrix elements of $U'(J)^{-1}$ by

$$(U^*_{J,M}(1.2), U_\alpha(1)\, U_{J-\frac{1}{2},\beta}(2)) = (\tfrac{1}{2}, \alpha, J-\tfrac{1}{2}, \beta \,|\, J, M)$$

and (6.5)

$$(U^*_{J-1,M}(1.2), U_\alpha(1)\, U_{J-\frac{1}{2},\beta}(2)) = (\tfrac{1}{2}, \alpha, J-\tfrac{1}{2}, \beta \,|\, J-1, M)$$

The generalized $\bar{\Sigma}(J)$ matrix is easily found by

$$\bar{\Sigma}(J) = U'(J)^{-1}\bar{\Sigma}^P(J)U'(J) \tag{6.6}$$

It is clear that the $U(2)$ theory is consistent with a theory of composite fields.

7. *An Application*

As an example, we have calculated angular distributions of the Poynting vector of fields defined by

$$\bar{N}(J) = \frac{c}{16\pi}(F^+(J)^\dagger, F^-(J)^\dagger)\begin{pmatrix} -\frac{1}{J}\bar{S}(J) & \bar{0} \\ \bar{0} & \frac{1}{J}\bar{S}(J) \end{pmatrix}\begin{pmatrix} F^+(J) \\ F^-(J) \end{pmatrix} \tag{7.1}$$

As was seen in Section 5, the total angular momentum $\bar{I}(J)$ is a constant of the motion. Therefore $F^+(J)$ and $F^-(J)$ fields for a given I $(I \geqslant J)$ and its z component M are expressed as

$$F^+_{I,M}(J,\mu) = \sum_{L=I-J}^{I+J} (L, M-\mu, J, \mu \,|\, I, M) g_L(r) Y_{L,M-\mu}(\Omega)$$

(7.2)

$$F^-_{I,M}(J,\mu) = \sum_{L=I-J}^{I+J} (L, M-\mu, J, \mu \,|\, I, M) f_L(r) Y_{L,M-\mu}(\Omega)$$

where μ represents the z component of spin J. For $g_L(r)$ and $f_L(r)$ we have used asymptotic forms

Results for $J = 0, \frac{1}{2}, 1, \frac{3}{2}, 2$ and $I = 0, \frac{1}{2}, 1, \frac{3}{2}, 2$ are shown in Table I and some of them are shown in Figures 1 and 2. As was expected, radiation patterns for a given (I, M) are very much different for different J. In the case of the $J = 2$ field, the radiation patterns depend on the M value (not $|M|$ value).

8. *Concluding Remarks*

In this paper, we presented an elementary theory for a field with any spin J. An essential idea of this theory is to find an expression for $U(2)$ generators in multidimensional spherical tensor space, that is, a generalized Pauli spin matrix,

TABLE I. Radiation patterns for fields. J, the spin of the field, is $0, \frac{1}{2}, 1, \frac{3}{2}$ and 2. I, the total angular momentum is $0, \frac{1}{2}, 1, \frac{3}{2}$ and 2, and M is its z component

		J				
I	M	0	$\frac{1}{2}$	1	$\frac{3}{2}$	2
0	0	$\frac{1}{4}\pi$				
$\frac{1}{2}$	$\pm\frac{1}{2}$		$\frac{1}{4}\pi$			
1	± 1	$\frac{3}{8}\pi \sin^2\theta$		$\frac{3}{16}\pi(1+\cos^2\theta)$		
	0	$\frac{3}{4}\pi \cos^2\theta$		$\frac{3}{8}\pi \sin^2\theta$		
$\frac{3}{2}$	$\pm\frac{3}{2}$		$\frac{3}{8}\pi \sin^2\theta$		$\frac{1}{8}\pi(1+3\cos^2\theta)$	
	$\pm\frac{1}{2}$		$\frac{1}{8}\pi(1+3\cos^2\theta)$		$\frac{3}{8}\pi \sin^2\theta$	
2	± 1	$\frac{15}{32}\pi \sin^4\theta$		$\frac{5}{16}\pi(1-\cos^4\theta)$		$\frac{5}{64}\pi(1\pm\cos\theta)^4$
	± 1	$\frac{15}{8}\pi \sin^2\theta \cos^2\theta$		$\frac{5}{16}\pi(1-3\cos^2\theta+4\cos^4\theta)$		$\frac{5}{16}\pi \sin^2\theta\,(1\pm\cos\theta)^2$
	0	$\frac{5}{16}\pi(3\cos^2\theta-1)^2$		$\frac{15}{8}\pi \sin^2\theta \cos^2\theta$		$\frac{15}{32}\pi \sin^4\theta$

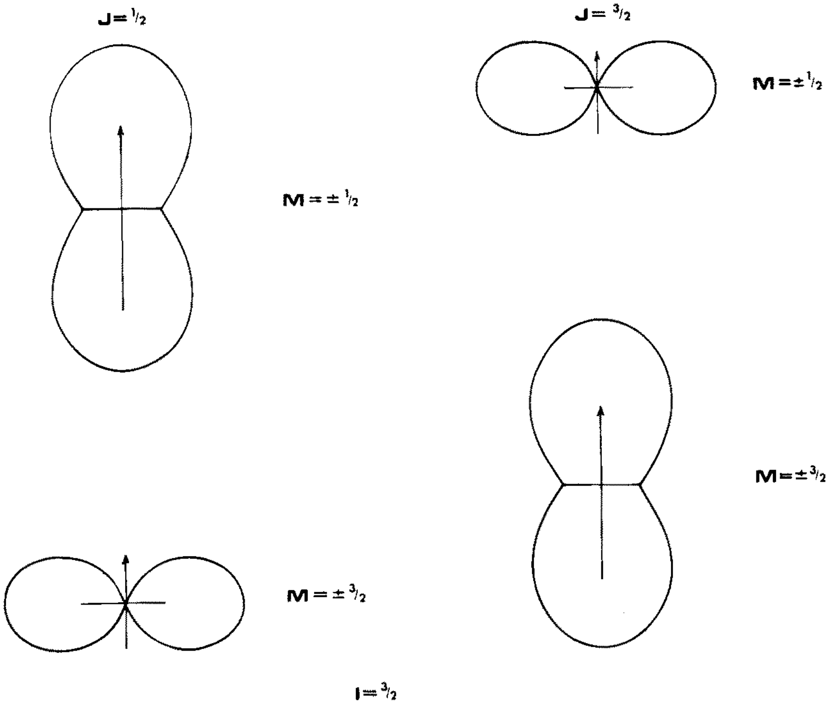

Figure 1–Radiation patterns for a field with spin $J = \frac{1}{2}$ and $\frac{3}{2}$. The arrow indicates the polar axis. I, the total angular momentum, is $\frac{3}{2}$ and M is its z component.

$\bar{\Sigma}(J)$. This is directly related to the generators of infinitesimal Lorentz transformations in multidimensional space, $\bar{A}(J)$ and $\bar{B}(J)$. This formalism led us to a concept of composite fields, and from it the field equations and the divergence conditions were naturally deduced. When the Hamiltonian was introduced we found that the total angular momentum is a constant of the motion. However, the fundamental constant of the motion is not the total angular momentum but $\bar{I}^p(J) = \bar{L} + \frac{1}{2}\hbar\bar{\Sigma}(J)$.

The modification of the present formalism for a nonvanishing mass will be considered in a future publication.

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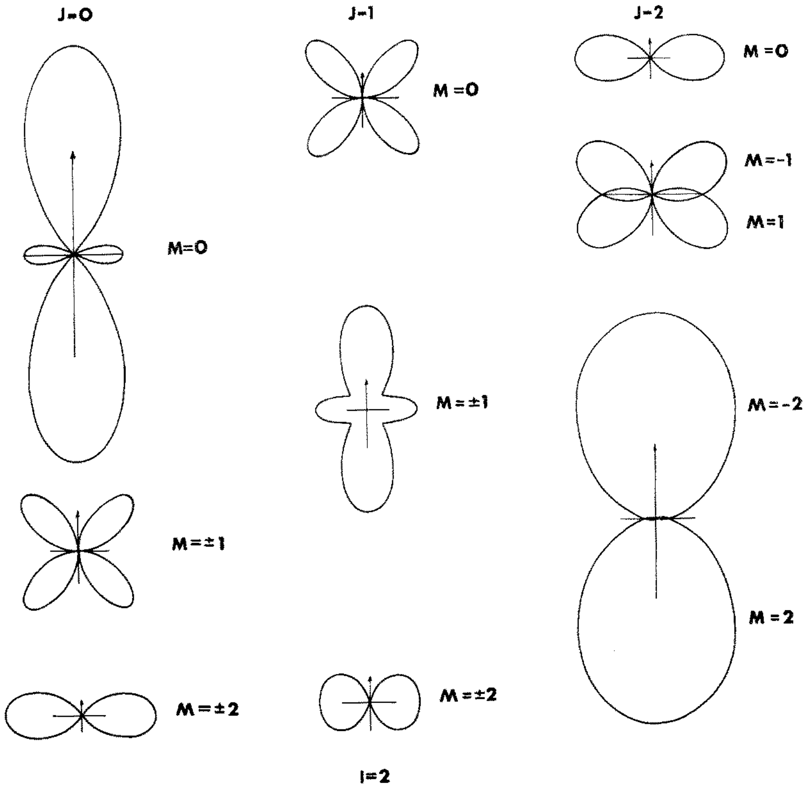

Figure 2—Radiation patterns for a field with spin $J = 0$, 1, and 2. The arrow indicates the polar axis. I, the total angular momentum is 2, and M is its z component.

Appendix

Define generalized space derivative operators as

$$\begin{aligned}
\text{curl}\,(J) &= -i\,\frac{1}{J}\,\bar{S}(J)\cdot\bar{\nabla} \\
\text{div}\,(J) &= -i\,\frac{1}{J}\,\bar{D}(J)\cdot\bar{\nabla} \\
\text{grad}\,(J) &= i\,\frac{1}{J}\,\bar{G}(J)\cdot\bar{\nabla} \\
\text{curl}'\,(J) &= -i\,\frac{1}{J}\,\bar{S}(J-1)\cdot\bar{\nabla}
\end{aligned} \tag{A1}$$

One can then prove the following relationships with the help of equations (3.6):

$$
\begin{aligned}
\operatorname{div}(J)\operatorname{grad}(J) - \operatorname{curl}'(J)\operatorname{curl}'(J) &= \mathbb{1}(J-1)\nabla^2 \\
\operatorname{curl}(J)\operatorname{grad}(J) - \operatorname{grad}(J)\operatorname{curl}'(J) &= \mathbb{O} \\
\operatorname{div}(J)\operatorname{curl}(J) - \operatorname{curl}'(J)\operatorname{div}(J) &= \mathbb{O}^+ \\
-\operatorname{curl}(J)\operatorname{curl}(J) + \operatorname{grad}(J)\operatorname{div}(J) &= \mathbb{1}(J)\nabla^2
\end{aligned}
\qquad \text{(A2)}
$$

where $\mathbb{O}$ is a $(2J+1) \times (2J-1)$ zero matrix and $\mathbb{O}^+$ is a $(2J-1) \times (2J+1)$ zero matrix.

These relationships are considered as a generalized vector algebra because if we set $J = 1$, $\operatorname{curl}'(J) = 0$, then one obtains the well-known vector algebra.

References

Good, Jr., R. H. (1957). *Physical Review,* **105**, 1914.
Hammer, C. L., and Good, Jr., R. H. (1957). *Physical Review,* **108**, 882.
Hammer, C. L., and Good, Jr., R. H. (1958). *Physical Review,* **1i1**, 342.
Laporte, O., and Uhlenbeck, G. E. (1931). *Physical Reveiw,* **37**, 1380.
Naimark, M. A. (1964). *Linear Representations of the Lorentz Group,* A Pergamon Press Book. (The MacMillan Company, New York), pp. 96 and 101.
Nelson, T. J., and Good, Jr., R. H. (1968). *Reviews of Modern Physics,* **40**, 508.
Shay, D., Song, H. S., and Good, Jr., R. H. (1965). *Nuovo Cimento Supplement,* **3**, 455.
Weaver, D. L., Hammer, C. L., and Good, Jr., R. H. (1964). *Physical Review,* **135**, B 241.